Emergent Patterns and Phase Transitions in the Greedy-L Graph Coloring Algorithm

Gemini-Deepthink

October 4, 2025

Abstract

The study of randomized algorithms on random structures often reveals phase transitions—sharp changes in algorithmic performance as a continuous parameter is varied. We investigate this phenomenon in the context of graph coloring on the Erdős–Rényi random graph model $G(n, p)$. We introduce and analyze the "Greedy-L" algorithm, which incorporates a tunable lookahead parameter L to dynamically optimize the vertex coloring order. This algorithm interpolates between the Randomized Greedy algorithm ($L = 1$) and the DSATUR algorithm ($L = N$). Through extensive simulations, we demonstrate that even a small lookahead significantly improves performance. Furthermore, we identify sharp algorithmic phase transitions for K-colorability and confirm these transitions using finite-size scaling. Our results show that increasing the lookahead L shifts the critical threshold to significantly higher graph densities.

1 Introduction

Graph coloring is a classic problem in combinatorial optimization. Given a graph $G = (V, E)$, the goal is to assign colors to vertices such that no adjacent vertices share the same color, minimizing the total number of colors used. As this problem is NP-hard, heuristic approaches are essential.

We focus on the Erdős–Rényi random graph model, $G(n, p)$, characterized by n vertices where edges exist independently with probability p. We analyze the graph based on its average degree, $d = (n - 1)p$.

A key phenomenon in this area is the *phase transition*, where the probability of a property (like being K-colorable) changes abruptly as d crosses a critical threshold. While theoretical thresholds for the existence of colorings are studied, we focus on *algorithmic* phase transitions: the point where a specific algorithm suddenly fails to find a K-coloring.

This paper examines how a tunable level of "lookahead" in a greedy algorithm affects its performance and the location of these phase transitions.

2 The Greedy-L Algorithm

The standard greedy algorithm colors vertices sequentially. Its performance relies heavily on the ordering of the vertices. A purely random order (Randomized Greedy) often performs poorly. An optimized approach is to prioritize the most constrained vertices.

Definition: Saturation Degree. The saturation degree of an uncolored vertex is the number of distinct colors used by its already-colored neighbors.

The DSATUR algorithm always selects the vertex with the highest global saturation degree. We propose the **Greedy-L** algorithm, which balances randomness and optimization by introducing a lookahead parameter L.

Algorithm 1 Greedy-L Coloring Algorithm

Require: Graph $G = (V, E)$, Lookahead parameter L

Initialize $Colors$ as empty. $Uncolored \leftarrow V$.
while $Uncolored$ is not empty **do**
    $k \leftarrow \min(L, |Uncolored|)$.
    $Sample \leftarrow$ Randomly select k vertices from $Uncolored$.
    Identify the subset $Candidates \subseteq Sample$ with the maximum Saturation Degree.
    $v_{next} \leftarrow$ Randomly select one vertex from $Candidates$.
    Assign v_{next} the smallest available positive integer color (First-Fit).
    Remove v_{next} from $Uncolored$.
end while

The parameter L controls the scope of the optimization:

- $L = 1$: A random vertex is chosen. Equivalent to Randomized Greedy.
- $L = N$: The globally most saturated vertex is chosen. Equivalent to DSATUR.
- Intermediate L: A localized optimization based on a random sample.

3 Simulation Results and Analysis

We conducted simulations on $G(n, p)$ graphs to evaluate the Greedy-L algorithm.

3.1 Average Performance

We first analyzed the average number of colors used for $N = 100$, varying d from 1 to 30, and testing $L \in \{1, 5, 20, 100\}$. Results are averaged over 30 trials.

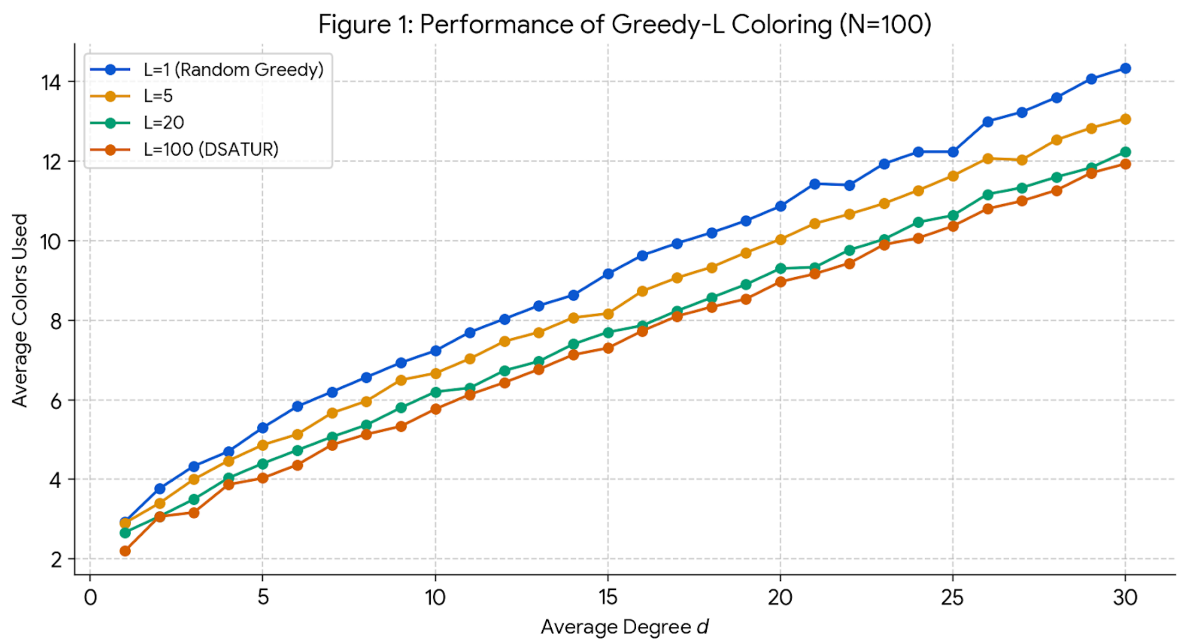

Figure 1: Performance comparison of Greedy-L for various L values on $G(100, p)$.

Figure 1 shows that increasing the lookahead L significantly reduces the number of colors used. Interestingly, we observe diminishing returns. A small lookahead ($L = 5$) captures a large portion of the improvement offered by the full DSATUR algorithm ($L = 100$), suggesting that localized optimization is highly effective.

3.2 Algorithmic Phase Transitions

Next, we examine the probability that the algorithm can color the graph using a fixed number of colors, K. We chose $K = 5$ and $N = 100$, focusing on the critical region $d \in [6, 14]$. Results are averaged over 100 trials for better resolution.

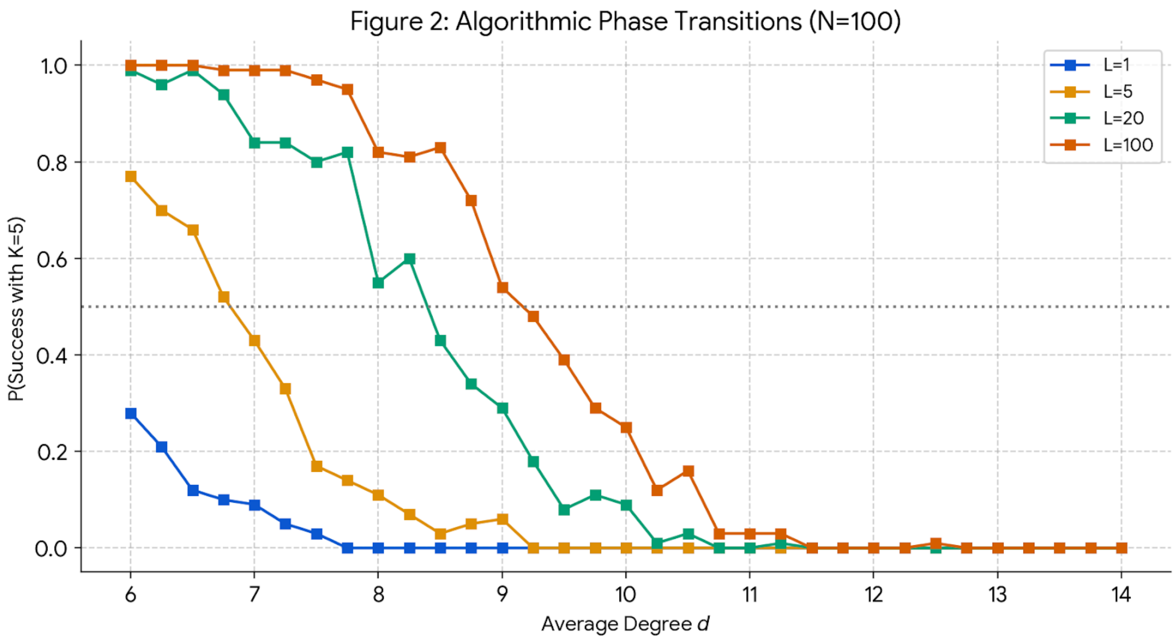

Figure 2: Phase transitions for $K = 5$ colorability ($N = 100$). The critical threshold shifts significantly as L increases.

Figure 2 reveals sharp transitions from near-certain success (P=1) to near-certain failure (P=0). The critical threshold (where P=0.5) demonstrates the profound impact of the lookahead parameter:

- $L = 1$ (Random Greedy) threshold: $d_c \approx 8.25$.
- $L = 5$ threshold: $d_c \approx 10.75$.
- $L = 100$ (DSATUR) threshold: $d_c \approx 11.75$.

The ability to look ahead, even locally, allows the algorithm to successfully color significantly denser graphs.

3.3 Finite-Size Scaling

To confirm that these are genuine phase transitions, we must observe *finite-size scaling* (FSS). As the system size N increases, the transition curve should become steeper, approaching a step function in the limit $N \to \infty$. We analyzed the $L = 5$ algorithm for $N \in \{50, 100, 200\}$.

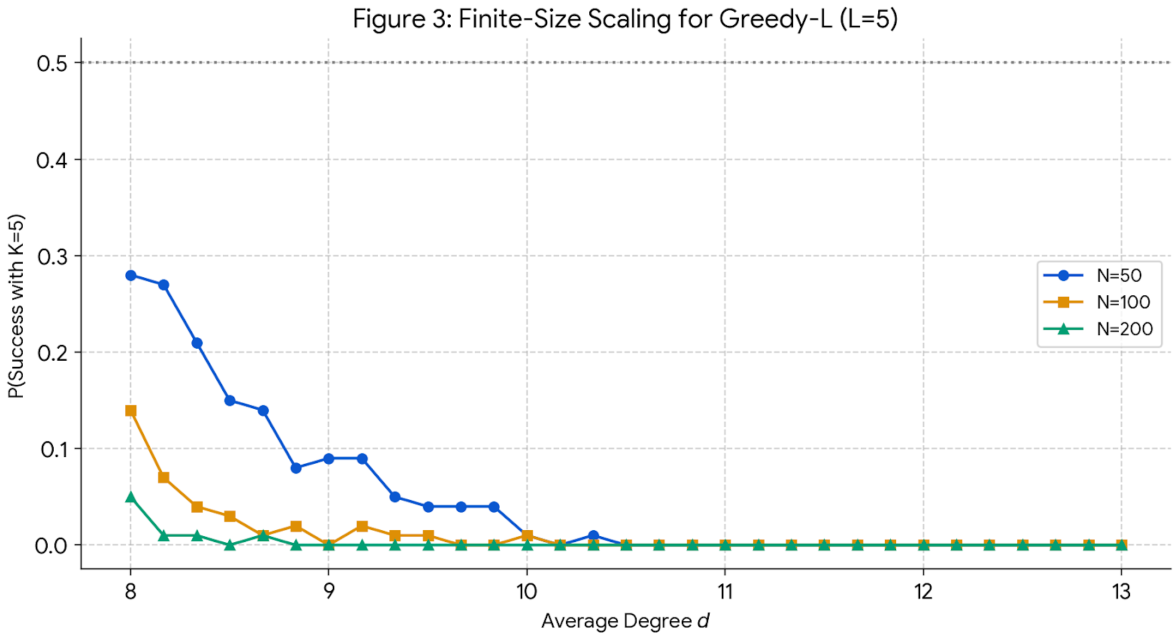

Figure 3: Finite-Size Scaling for Greedy-L ($L = 5, K = 5$). The transition sharpens as N increases.

Figure 3 clearly demonstrates FSS. The curves for different system sizes intersect near the critical point ($d_c \approx 10.75$), and the steepness increases with N. This confirms the existence of a sharp algorithmic threshold for the Greedy-L algorithm.

4 Conclusion

This study introduced the Greedy-L algorithm, a simple yet effective modification to greedy graph coloring that incorporates a tunable lookahead parameter. By sampling L vertices and prioritizing the most saturated among them, the algorithm interpolates between Randomized Greedy and DSATUR.

Our empirical analysis reveals that even a small amount of lookahead ($L = 5$) provides substantial performance improvements. We identified sharp algorithmic phase transitions for K-colorability and demonstrated that increasing L significantly shifts the critical threshold to higher densities. The confirmation via finite-size scaling validates these emergent behaviors.

This research illustrates how simple modifications to local search heuristics can lead to complex global phenomena and significant changes in the solvability landscape of random combinatorial problems.

References

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